Functions, Limits, and Continuity

1. Describe the level sets of the following functions. What shape are they?

   (a) $f(x, y) = x^2 + 4y^2$.

   (b) $f(x, y, z) = x^2 + 4y^2 + 9z^2$.

   (c) $f(x, y) = y - x$.

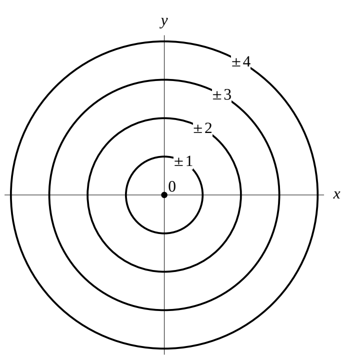
   (d) $f(x, y, z) = 2x + 3y + 4z$.

   (e) $f(x, y, z) = 4x^2 + 9y^2$.

2. Let S be the unit sphere centered at $(0, 0, 0)$. Is S the graph of a function? If so, what function?

   Is S a level set of a function? If so, what function?

3. Is the following picture the <u>level set diagram</u> (also known as <u>contour map</u>) of a function? If so, sketch the graph of the function.

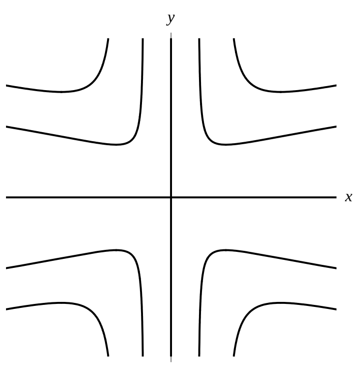


4. Match each function with its level set diagram and its graph. (Note that each function is undefined at $(0,0)$.)

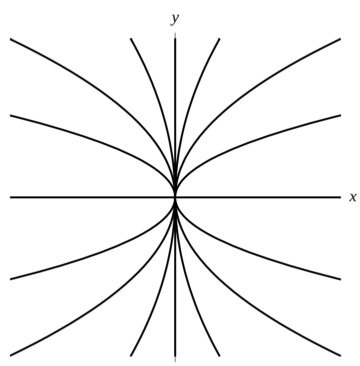
   (a) $f(x,y) = \dfrac{y^2}{x^2+y^2}$. (Hint: What are the level sets $f(x,y)=0$, $f(x,y)=\frac{1}{2}$, and $f(x,y)=1$?)

   (b) $f(x,y) = -\dfrac{xy^2}{x^2+y^4}$. (Hint: What are the level sets $f(x,y)=\frac{1}{2}$ and $f(x,y)=-\frac{1}{2}$?)

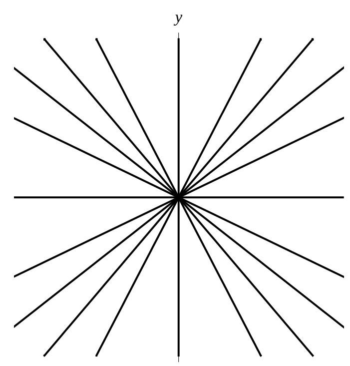
   (c) $f(x,y) = -\dfrac{xy^2}{x^2+y^2}$. (Hint: Process of elimination!)



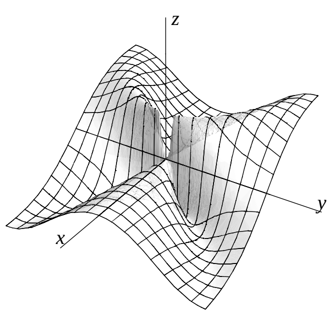
(I)



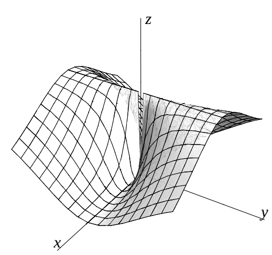
(II)



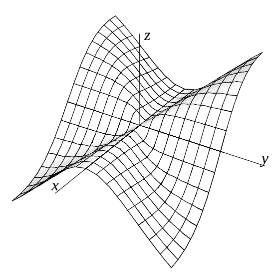
(III)



(A)



(B)



(C)

Definition. The limit of $f(x,y)$ as (x,y) approaches (a,b) is L if we can make the values of $f(x,y)$ as close to L as we like by taking the point (x,y) sufficiently close to the point (a,b), but not equal to (a,b). We write this as $\lim_{(x,y)\to(a,b)} f(x,y) = L$.

Strategy.

- To show that a limit $\lim_{(x,y)\to(a,b)} f(x,y)$ does *not* exist, we usually try to find two different paths approaching (a,b) on which $f(x,y)$ has different limits.
- Showing that a limit $\lim_{(x,y)\to(a,b)} f(x,y)$ *does* exist is generally harder. If the point (a,b) is $(0,0)$, one strategy is to rewrite the limit in polar coordinates. Then, no matter how (x,y) approaches $(0,0)$, r tends to 0, so if the limit $\lim_{r\to 0^+} f(r\cos\theta, r\sin\theta)$ exists, then the original limit $\lim_{(x,y)\to(0,0)} f(x,y)$ also exists.

5. Using the contour maps from #4, first guess whether $\lim_{(x,y)\to(0,0)} f(x,y)$ exists for each of the following functions. Then show that your guess is correct using the strategy described above.

   (a) $f(x,y) = \dfrac{y^2}{x^2+y^2}$.

   (b) $f(x,y) = -\dfrac{xy^2}{x^2+y^4}$.

   (c) $f(x,y) = -\dfrac{xy^2}{x^2+y^2}$.